EXMIK[X]

$\boxed{9/2}$

$$P(X) = \left(\sum_{k=0}^{n-1} X^k\right)^2 - n^2 X^{n-1} \qquad n \geq 2$$

Mq 1 est racine $\geq$ double de P ie

$$\begin{cases} P(1) = 0 \\ P'(1) = 0 \end{cases}$$

Or $P' = 2\sum_{k=0}^{n-1} kX^{k-1} \sum_{k=0}^{n-1} X^k - n^2(n-1)X^{n-2}$

$$P(1) = \left(\sum_{k=0}^{n-1} 1\right)^2 - n^2 \times 1 = n^2 - n^2 = 0$$

$$P'(1) = 2\sum_{k=0}^{n-1} k \sum_{k=0}^{n-1} 1 - n^2(n-1)$$

$$= 2\,\frac{n(n-1)}{2}\,n - n^2(n-1)$$

$$= 0$$

On a bien montré que $(X-1)^2 \mid P$ ie 1 est racine double

Est-elle triple ?

$$P'' = 2\left(\sum_{k=0}^{n-1} k(k-1)X^{k-2}\sum_{k=0}^{n-1} X^{k} + \left(\sum_{k=0}^{n-1} kX^{k-1}\right)^2\right)$$

$$-\, n^2(n-1)(n-2)X^{n-3}$$

$$P''(1) = 2\left(\left(\sum_{k=0}^{n-1} k^2 - \sum_{k=0}^{n-1}\right)n + \left(\sum_{k=0}^{n-1} k\right)^2\right) - n^2(n-1)(n-2)$$

$$= 2\left(\left(\frac{n(n-1)(2n-1)}{6} - \frac{n(n-1)}{2}\right)n + \left(\frac{n(n-1)}{2}\right)^2\right)$$

$$-\, n^2(n-1)(n-2)$$

$$= n^2(n-1)\left[\frac{2n-1}{3} - 1 + \frac{n-1}{2} - (n-2)\right]$$

$$= n^2(n-1)\,\frac{n+1}{6}$$

$$\neq 0 \quad \text{pour } n \geq 2$$

donc 1 n'est pas racine triple ie 1 est racine double.

$\boxed{9/1}$

$$P = (X+1)^{2n+1} + X^{n+2} \overset{?}{\Big|} X^2+X+1$$

$$X^2+X+1 = (X-j)(X-\bar{j})$$

$$\bar{j} = \frac{-1-\sqrt{3}i}{2} \qquad j = \frac{-1+\sqrt{3}i}{2}$$

Ainsi, $X^2+X+1 \mid P \iff \begin{cases} P(j)=0 \\ P(\bar{j})=0 \end{cases}$

$\iff P(j) = 0$

car $P \in \mathbb{R}[X]$ donc $P(\alpha)=0$
$\iff P(\bar{\alpha})=0$

$$P(j) = (j+1)^{2n+1} + j^{n+1}$$

Or $\begin{cases} j^2+j+1=0 \\ \text{donc } j^2 = 1 \end{cases}$

donc $P(j) = (-j^2)^{2n+1} + j^{n+2}$

$= (-1)^{2n+1} j^{4n+2} + j^{n+2}$

$= -j^{4n+2} + j^{n+2}$

$= -j^{3n+n+2} + j^{n+2} = -(j^3)^n j^{n+2} + j^{n+2} = 0$

$\boxed{8/2}$

$$P_n = 1 + \frac{X}{1!} + \dots + \frac{X^n}{n!}$$

$$P_n' = \underbrace{\frac{1}{1!}}_{1} + \underbrace{\frac{2X}{2!}}_{\frac{X}{1!}} + \dots + \frac{X^{n-1}}{(n-1)!}$$

$$= P_n - \frac{X^n}{n!}$$

Mq il n'y a pas de racines multiples par l'absurde

Notons α une racine

$$\begin{cases} P_n(\alpha) = 0 \\ P_n'(\alpha) = 0 \end{cases} \Leftrightarrow \begin{cases} P_n(\alpha) = 0 \\ P_n(\alpha) - \frac{\alpha^n}{n!} = 0 \end{cases}$$

$$\Rightarrow \frac{\alpha^n}{n!} = 0$$

$$\Leftrightarrow \alpha = 0$$

Or $P_n(0) = 1 \neq 0$ donc il n'y a pas de racine multiple $\boxed{\lightning}$

$\boxed{4/1}$

Sans les relations coefs-racines

$$\sum_{\omega \in \mathbb{U}_n} \omega = \sum_{k=0}^{n-1} e^{\frac{2ik\pi}{n}} = \begin{cases} \dfrac{e^{2i\pi} - 1}{e^{2i\pi/n} - 1} & \text{si } n \neq 1 \\ 1 & \text{si } n = 1 \end{cases}$$

$$\boxed{\mathbb{U}_n = \{z \in \mathbb{C},\ z^n = 1\} = \{e^{\frac{2ik\pi}{n}},\ k \in [\![0, n-1]\!]\}}$$

$$= \begin{cases} 0 & \text{si } n \neq 1 \\ 1 & \text{si } n = 1 \end{cases}$$

Meth 1 À la El Baki

$$\prod_{\omega \in \mathbb{U}_n} \omega = \begin{cases} -1 & \text{si } n \in 2\mathbb{N} \\ 1 & \text{si } n \in 2\mathbb{N}+1 \end{cases} = (-1)^{n+1}$$

car les racines $\cancel{\mathbb{R}}$ sont conjuguées deux à deux et $\omega\bar{\omega} = |\omega|^2 = 1$ et les racines $\mathbb{R}$ sont

$$\begin{cases} 1 \text{ et } -1 & \text{si } n \in 2\mathbb{N} \\ 1 & \text{si } n \in 2\mathbb{N}+1 \end{cases}$$

Meth 2

$$\prod_{\omega \in \mathbb{U}_n} \omega = \prod_{k=0}^{n-1} e^{\frac{2ik\pi}{n}} = \exp\left(\sum_{k=0}^{n-1} \frac{2ik\pi}{n}\right) = \frac{2i\pi}{n} \exp\left(\sum_{k=0}^{n-1} k\right)$$

$$= \exp\left(\frac{2i\pi}{n} \frac{n(n-1)}{2}\right)$$

$$= (e^{i\pi})^{n-1} = (-1)^{n-1}$$